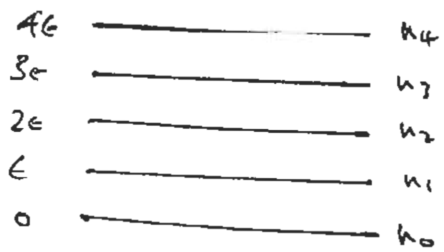
PHYS 393. January 2007. Solutions

1a.



Distinguishable particles

Number particles $N = 5$

Total energy $U = 4\epsilon$.

(i)

2 distributions

Distributions	0	ϵ	2ϵ	3ϵ	4ϵ	(ii) No microstates.	2 microstates
1.	4	–	–	–	1	5	5
2.	3	1	–	1	–	20	20
3.	3	–	2	–	–	${}^5C_2 = \frac{5\times4}{1\times2} = 10$	10
4.	2	2	1	–	–	$5\times{}^4C_2 = 5\times\frac{4\times3}{1\times2} = 30$	30
5.	1	4	–	–	–	5	5

Total number microstates = 70.

(iii) Evaluate Population

$$n_0 = \frac{5}{70}\times4 + \frac{20}{70}\times3 + \frac{10}{70}\times3 + \frac{30}{70}\times2 + \frac{5}{70}\times1 = \frac{175}{70} = 2.50$$

$$n_1 = \frac{20}{70}\times1 + \frac{30}{70}\times2 + \frac{5}{70}\times4 = \frac{100}{70} = 1.43$$

$$n_2 = \frac{10}{70}\times2 + \frac{30}{70}\times1 = \frac{50}{70} = 0.72$$

$$n_3 = \frac{20}{70} = 0.29$$

$$n_4 = \frac{5}{70} = 0.07$$

Populations (distinguishable) 2

(iv) For bosons 1 microstate per distribut^n

See next sheet for populations.

1(a.) contd.

(iv) Bosons – 1 microstate / distribution 5 microstates in total

$$n_0 = \frac{1}{5}\{4 + 3 + 3 + 2 + 1\} = 13/5 = 2.6$$

$$n_1 = \frac{1}{5}\{1 + 2 + 4\} \quad 1.40$$

$$n_2 = \frac{1}{5}\{1 + 2\} = 0.6$$

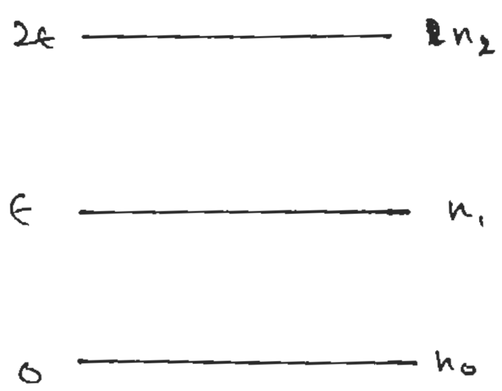
$$n_3 = \frac{1}{5} = 0.2$$

$$n_4 = 1/5 = 0.2$$

Population (indistinguishable)

2

1 (b)



(i) $n_1 = n_0 \exp(-\epsilon/kT)$

$0.1 = \exp\left(-\frac{1.38\times10^{-21}}{1.38\times10^{-23}\,T}\right)$

$\ln(0.1) = -2.30 = -\frac{100}{T}$ $\qquad T = \frac{100}{2.3} = 43.5\,K$ 2

(ii) $n_2 = n_0 \exp(-2\epsilon/kT)$

$10^{-6} = \exp\left(-\frac{2\times1.38\times10^{-21}}{1.38\times10^{-23}\,T}\right)$

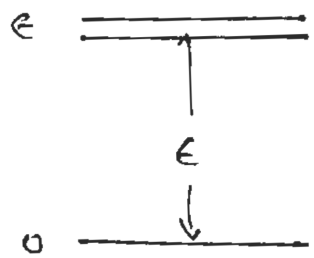
$\ln(10^{-6}) = -13.82 = -\frac{200}{T}$ $\qquad T = +\frac{200}{13.82} = 14.5\,K$ 2

(iii) $n_2 = n_1 \exp(-\epsilon/kT)$

$0.99 = \exp\left(-\frac{1.38\times10^{-24}}{1.38\times10^{-23}\,T}\right)$

$\ln(0.99) = -0.0100 = \frac{100}{T}$ $\qquad T = \frac{100}{0.01} = 10^4\,K$ 2

1(c)



(i) Partition function Z

$$Z = 1 + 2\exp(-\epsilon/kT)$$ 2

(ii) $$U = NkT^2 \frac{\partial}{\partial T}(\ln Z) = NkT^2 \frac{\partial}{\partial T} \ln\left[1 + 2\exp(-\epsilon/kT)\right]$$

$$U = NkT^2 \left\{ \frac{2 \cdot (\epsilon/kT^2)\exp(-\epsilon/kT)}{\left[1 + 2\exp(-\epsilon/kT)\right]} \right\}$$

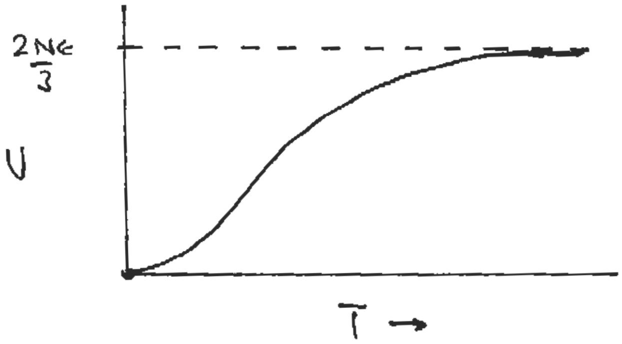
$$= \frac{2N\epsilon \exp(-\epsilon/kT)}{\left[1 + 2\exp(-\epsilon/kT)\right]}$$ 2

(iii) Limits

As $T \to 0$ $\quad \exp(-\epsilon/kT) \to 0 \quad U \to 0$ 1

As $T \to \infty$ $\quad \exp(-\epsilon/kT) \to 1 \quad U \to \frac{2N\epsilon}{3}$ 1

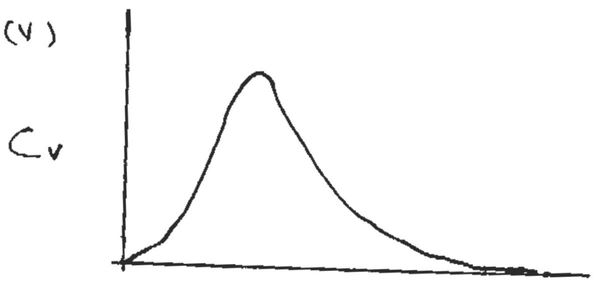
(iv)



2

$C_V = \left(\frac{\partial U}{\partial T}\right)$

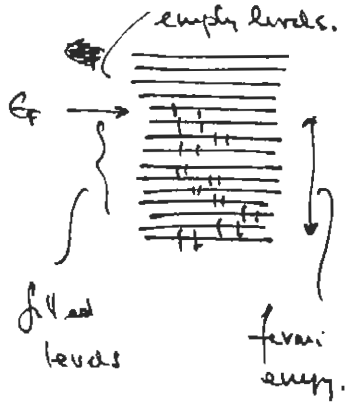
(v)



C_V is slope of U vs T

2

1(d) Fermi energy of a system of electrons.

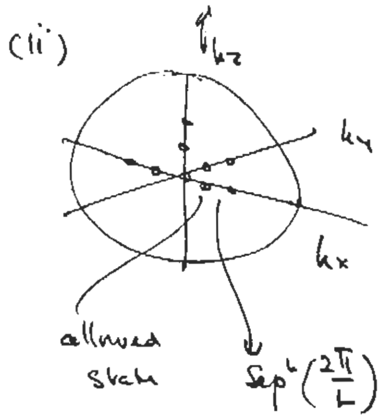


When N electrons populate set of quantized states
– they obey Pauli principle.
~~States~~ Only ↑ and ↓ electrons in each state.
Filling these states with N electrons – fills them up to certain level of energy E_F.

This E_F is fermi energy.

(i) 2 arises because of ~~double~~ population of k state with ↑ and ↓ electrons.

(ii)



Number of k states up to k_F

$$= \frac{4}{3}\pi k_F^3 \cdot \frac{1}{\left(\frac{2\pi}{L}\right)^3}$$

↑↓

Population of electrons $N = 2 \times \frac{4}{3}\pi k_F^3 \cdot \frac{V}{(2\pi)^3}$

$$k_F^3 = \frac{3N\pi^2}{V} \quad ; \quad k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$$

(iii) $$\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$$

(iv) Na. 23 kg Na contains 6×10^{26} Na atoms

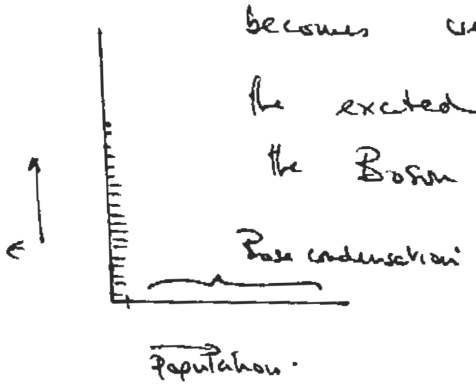
$1\,m^3 \equiv 971$ kg Na contains $\frac{6.02 \times 10^{26}}{23} \times 971 = 2.54 \times 10^{28}$ atoms.

$\frac{N}{V} = 2.54 \times 10^{28}$.

$$\epsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 \times 2.54 \times 10^{28}\right)^{2/3} = \left(\frac{6.63 \times 10^{-34}}{2\pi}\right)^2 \cdot \frac{1}{2 \times 9.11 \times 10^{-31}}\left(3\pi^2 \times 2.54 \times 10^{28}\right)^{2/3}$$

1(a). Bose condensation.

Below a critical temperature T_B in a boson system - the population of the ground state of a series of quantised states becomes very large and discontinuous with the population of the excited states. This extra population (non-continuous) is the Boson condensation



(ii) $$T_B = \left(\frac{h^2}{2\pi m k}\right)\left(\frac{N}{2.612 V}\right)^{2/3}$$

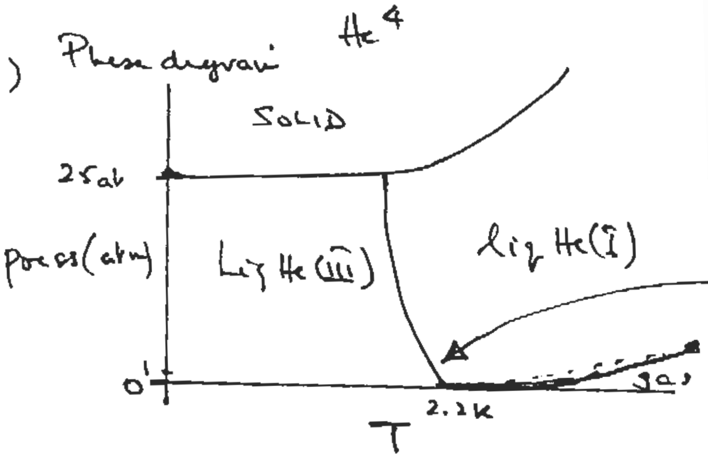
For He^4 liquid ~~[illegible] 27×10~~ 6×10^{23} atoms contained in 27×10^{-6} m^3.

In 1 m^3 there are $\frac{6.02\times10^{23}}{27\times10^{-6}} = 2.23\times10^{28} = \frac{N}{V}$.

$$T_B = \left(\frac{(6.63\times10^{-34})^2}{2\pi\times4\times1.67\times10^{-27}\times1.38\times10^{-23}}\right)\left(\frac{2.23\times10^{28}}{2.612}\right)^{2/3}$$

$$T_B = 3.2\ K$$

(iii) Phase diagram He^4

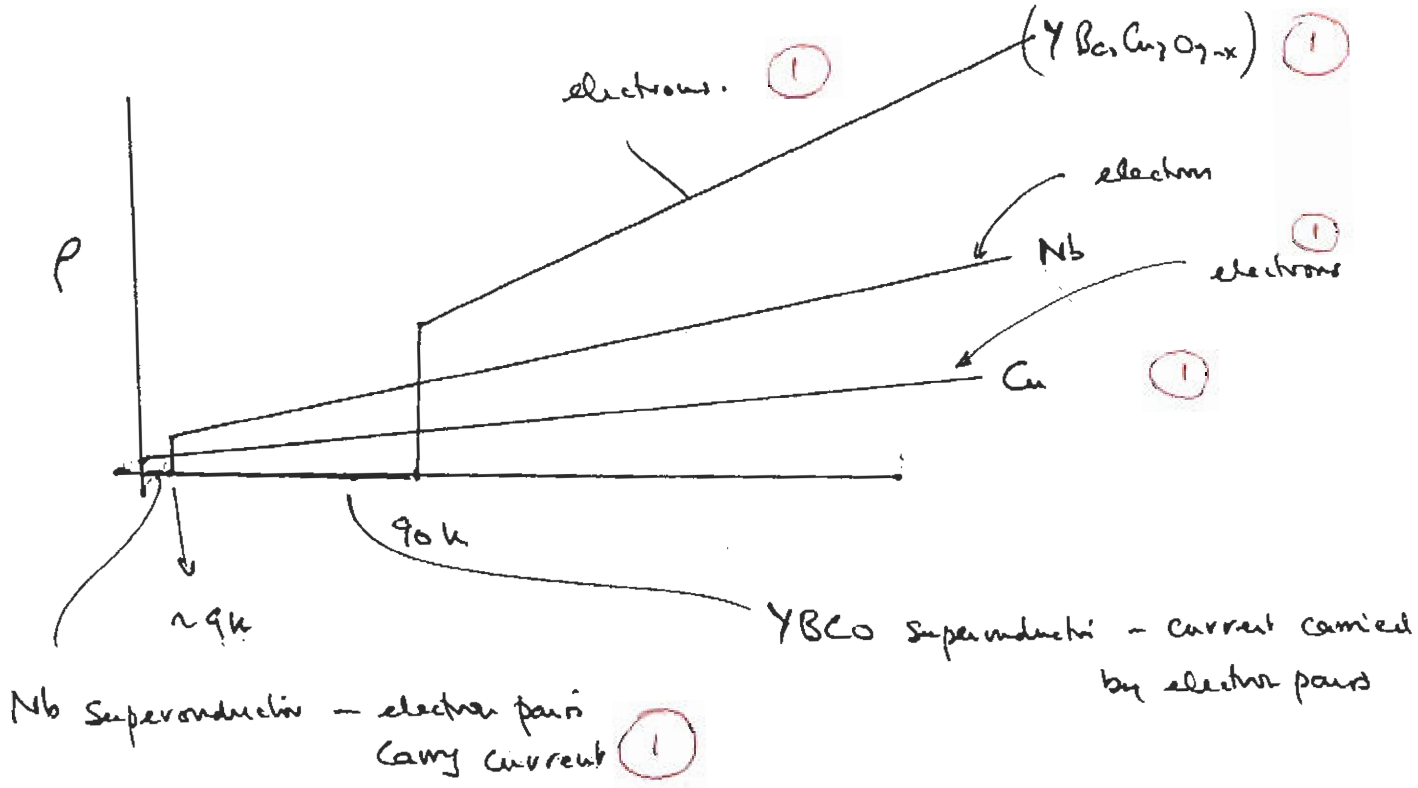


(iv) Bose condensation associated with phase change Liq He (I) to Liq He (II) — the λ line.

1(f). Current carriers.

(i) In superconductor current carriers are electron pairs. 1

(ii) In normal conductor current carriers are single electrons 1



2(a) $k = \frac{2\pi}{\lambda} \qquad \lambda = \frac{h}{mv} \qquad k = \frac{2\pi mv}{h}$

$$v = \left(\frac{h}{2\pi}\right)\frac{k}{m} = \frac{\hbar k}{m} \qquad 1$$

$$\epsilon = \frac{1}{2} m v^2 \qquad 1$$

$$\epsilon = \frac{1}{2} m . \frac{\hbar^2 k^2}{m^2} = \frac{\hbar^2 k^2}{2m} \qquad 1$$

(ii) $$g(k)\,dk = \frac{4\pi V k^2 \,dk}{(2\pi)^3}$$

$$k = \frac{mv}{\hbar} \qquad dk = \left(\frac{m}{\hbar}\right) dv \qquad 1$$

$$g(v)\,dv = V . \frac{4\pi}{(2\pi)^3} . \frac{m^2 v^2}{\hbar^2} . \frac{m}{\hbar} . dv \qquad 1$$

$$g(v)\,dv = 4\pi V \left(\frac{m}{h}\right)^3 v^2 \,dv \qquad 1$$

(iii) $$Z = \int_0^\infty g(k)\,dk \,\exp\left(-\frac{\hbar^2 k^2}{2mkT}\right)$$

$$= \int_0^\infty \frac{4\pi V k^2 \,dk}{(2\pi)^3} . \exp\left(-\frac{\hbar^2 k^2}{2mkT}\right) \qquad 1$$

I_2 with $b = \frac{\hbar^2}{2mkT}$.

$$Z = \frac{4\pi}{(2\pi)^3} . V . \frac{1}{2\left(\hbar^2/2mkT\right)} . \frac{1}{2}\left(\frac{\pi}{\hbar^2/2mkT}\right)^{1/2} \qquad 1$$

$$= \frac{4\pi V}{(2\pi)^3} . \frac{(2mkT)}{2\hbar^2} . \frac{1}{2} . \frac{(2mkT.\pi)^{1/2}}{\hbar}$$

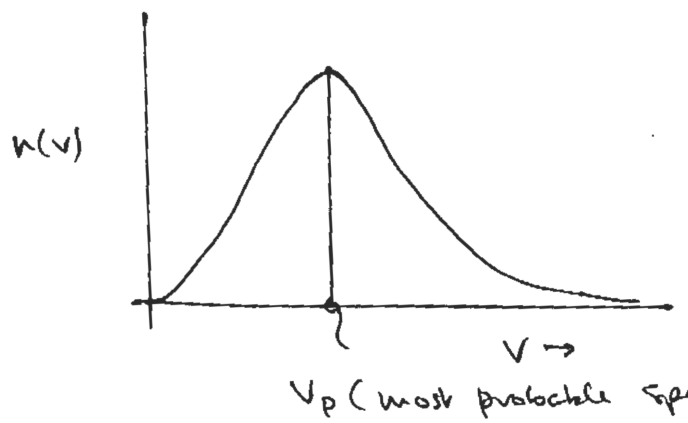
$$Z = \frac{\cancel{4}\pi V}{\cancel{4}} . \frac{(2mkT)}{h^2} . \frac{(2\pi mkT)^{1/2}}{h} = V . \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \qquad 1$$

(iv) Distribution of speeds.

$$n(v)\,dv = 4\pi . V . \left(\frac{m}{h}\right)^3 v^2\,dv . \frac{N}{Z} \exp\left(-mv^2/2kT\right). \qquad (1)$$

$$= 4\pi \cancel{V} \frac{. N}{\cancel{V}.} \left(\frac{m}{\cancel{h}}\right)^3 . v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv . \frac{\cancel{h^3}}{(2\pi m k T)^{3/2}} \qquad (1)$$

$$= 4\pi N . \left(\frac{m}{2\pi k T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv \qquad (1)$$



(2)

(vi) Mean ~~square~~ speed.

$$b = +\frac{m}{2kT}.$$

$$\langle v^2 \rangle = \frac{\int_0^\infty v^2 n(v)\,dv}{\int_0^\infty n(v)\,dv} = \frac{\text{consts}}{\text{consts}} \frac{\int_0^\infty v^4 \exp\left(-\frac{mv^2}{2kT}\right) dv}{\int_0^\infty v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv} \qquad (1)$$

$$\langle v^2 \rangle = \frac{I_4}{I_2} = \frac{(n-1)}{2b}^{\,n=4} = \frac{3}{2\left(m/2kT\right)} = \frac{3kT}{m}. \qquad (1)$$

$$\langle v^2 \rangle_{\text{He at } 300\,\text{k}} = \frac{3 \times 1.38 \times 10^{-23} \times 300}{4 \times 1.66 \times 10^{-27}} = 1.87 \times 10^{6}\ \text{m}^2/\text{s}^2 \qquad (1)$$

(vii) Energy / mole $U = N . \frac{1}{2} m \langle v^2 \rangle$

$$= \frac{N \cancel{m}}{2} . \frac{3kT}{\cancel{m}} = \frac{3}{2} NkT.$$ 1

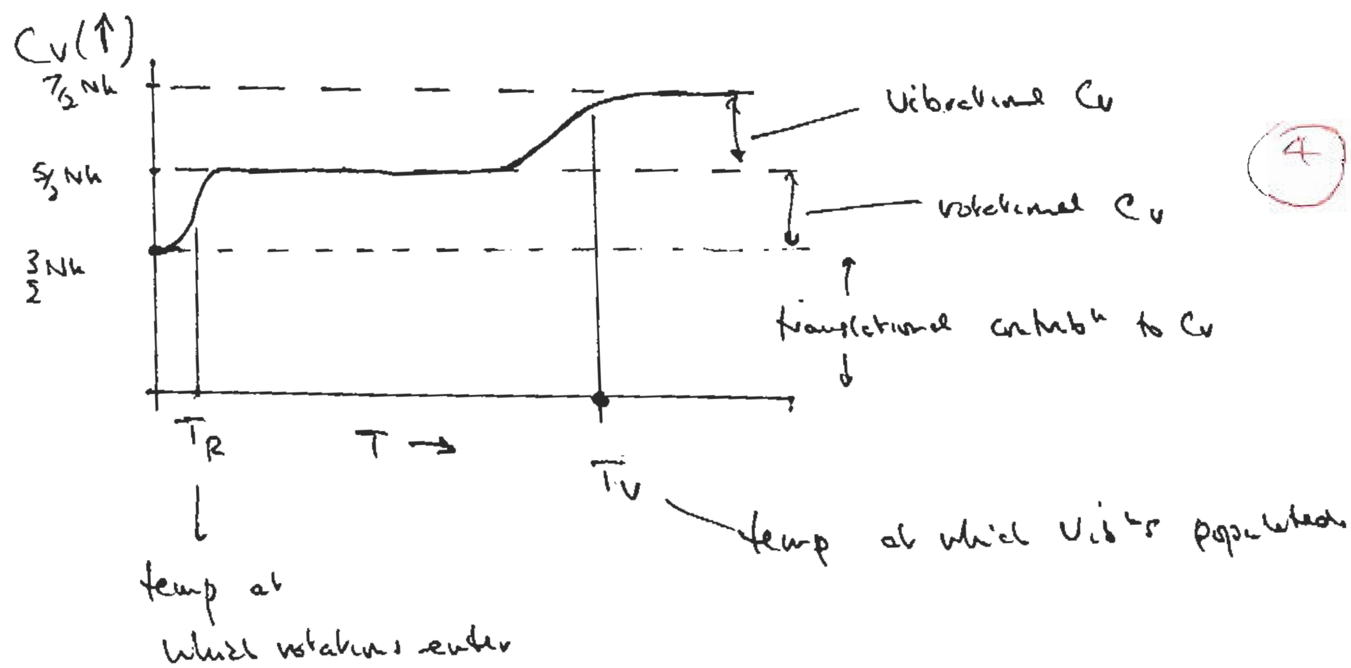
$$C_v / \text{mole} = \left(\frac{\partial U}{\partial T}\right) = \frac{3}{2} Nk.$$ 1

(viii) Additional excitations in diatomic gas.

Vibration 1

Rotation 1

(ix)



4

Thus as $T \to \infty$

$$U \to \frac{\overset{3}{\cancel{12}}\pi \cancel{V} k^{\cancel{4}} T^{\cancel{4}}}{\cancel{c^3}\cancel{h^3}} \cdot \frac{\cancel{h^3} \cdot \cancel{3}\cancel{c^3}}{k^{\cancel{3}} T^{\cancel{3}} \cdot \cancel{3}} \cdot \frac{N}{\cancel{V}} \cdot \frac{1}{\cancel{4\pi}}$$

1

$$U \to 3NkT.$$

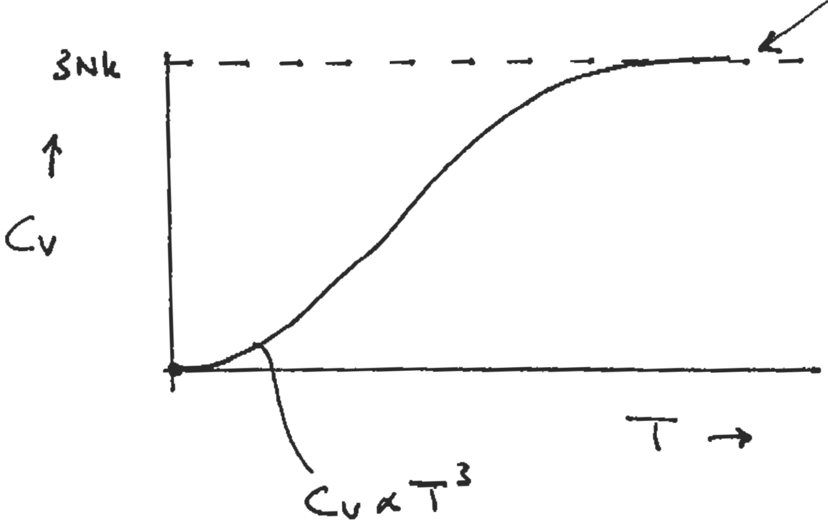
(viii) $$C_v = \left(\frac{\partial U}{\partial T}\right)_v$$

As $T \to 0$ $$C_v \to \frac{\partial}{\partial T}\left(\frac{4}{5}\pi^5 \frac{V k^4 T^4}{c^3 h^3}\right)$$

$$C_v \to \frac{16\pi^5}{5} V \left(\frac{k}{hc}\right)^3 k T^3 .$$ 1

As $T \to 0$ $C_v \to 3Nk.$ 1

(ix)



4

As $T \to 0$ $C_v \to 0$ as $C_v = \alpha T^3$ — good agreement with data

$T \to \infty$ $C_v \to 3Nk$ — agrees with expt.

Between — shape quite good but not exact.

Overall — good simple theory for C_v vs T

2 (b) Conditions for quantised phonons

(i) $k_x = \frac{2\pi n_x}{L}$, $k_y = \frac{2\pi n_y}{L}$ $k_z = \frac{2\pi n_z}{L}$ (1)

where $n_x, n_y, n_z = \pm 1, \pm 2, \pm 3 \ldots$

(ii) $g(k)\,dk = \frac{\text{volume of } k \text{ space}}{\text{space / allowed state}} = \frac{4\pi k^2\,dk}{(2\pi/L)^3}$ (1) (1)

$$g(k)\,dk = \frac{4\pi . V . k^2\,dk}{(2\pi)^3}$$

Taking account of longitudinal ⟷ and transverse ↑↗ modes.

$$g(k) = 3 \times \frac{4\pi V k^2 dk}{(2\pi)^3}$$ (1)

(iii) $\nu = \frac{kc}{2\pi}$ $k = \frac{2\pi\nu}{c}$ $dk = \frac{2\pi}{c}\,d\nu$ (1)

$$g(\nu)\,d\nu = 3 . \frac{4\pi V .}{\cancel{(2\pi)^3}} \left(\frac{\cancel{2\pi}\nu}{c}\right)^2 \left(\frac{\cancel{2\pi}}{c}\right) d\nu$$ (1)

$$= \frac{12\pi V \nu^2\,d\nu}{c^3} .$$ (1)

For cut off frequency

$$3N = \int_0^{\nu_D} g(\nu)\,d\nu = \int_0^{\nu_D} \frac{12\pi V}{c^3} \nu^2\,d\nu = \frac{12\pi V}{c^3} \frac{\nu_D^3}{3}$$ (1) (1)

$$\nu_D^3 = \frac{3Nc^3}{4\pi V}$$ (1)

(iv) $h\nu$ = quantum of phonon energy (1)

$\frac{1}{[\exp(h\nu/kT)-1]}$ = boson probability of occupation of state of energy $h\nu$ at temperature T. (1)

(v) Internal energy $U = \int_0^{\nu_D} \frac{12\pi V}{c^3} . \nu^2 d\nu . h\nu . \frac{1}{[\exp(h\nu/kT)-1]}$ (1)

Put $y = h\nu/kT$ $\quad y_D = \frac{h\nu_D}{kT}$.

$dy = \frac{h}{kT} d\nu$ or $d\nu = \frac{kT\,dy}{h}$ (1)

Thus $U = \frac{12\pi V h}{c^3} \int_0^{y_D} \frac{\left(\frac{kT}{h}\right)^3 y^3 \left(\frac{kT}{h}\right) dy}{[\exp(y)-1]}$

$$U = \frac{12\pi V}{c^3} \frac{k^4 T^4}{h^3} \int_0^{y_D} \frac{y^3\,dy}{[\exp(y)-1]}$$ (1)

(vi) As $T \to 0$ $\quad y \to \infty$

thus $\int_0^{y_D} \frac{y^3\,dy}{[\exp(y)-1]} \to \int_0^{\infty} \frac{y^3\,dy}{[\exp(y)-1]} = \pi^4/15$ (1)

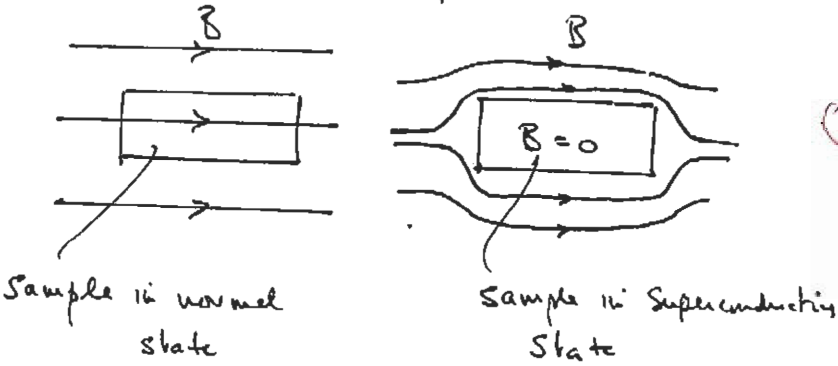
Thus $U \to \frac{12\pi V}{c^3} \frac{k^4 T^4}{h^3} . \frac{\pi^4}{15} = \frac{4\pi^5 V k^4 T^4}{5\, c^3 h^3}$ (1)

(vii) As $kT \gg h\nu$ $\quad y \to 0$ $\quad \int_0^{y_D} \frac{y^3\,dy}{[\exp(y)-1]} \to \int_0^{y_D} \frac{y^{\cancel{3}\,2}\,dy}{[1+\cancel{y}..-1]} = \frac{y_D^3}{3}$ (1)

Then $U \to \frac{12\pi V k^4 T^4}{c^3 h^3} . \frac{1}{3} \left(\frac{h^3 \nu_D^3}{k^3 T^3} \right)$ but $\nu_D^3 = \frac{3c^3}{} . N$

3(a)

(i) Meissner effect in superconductors.



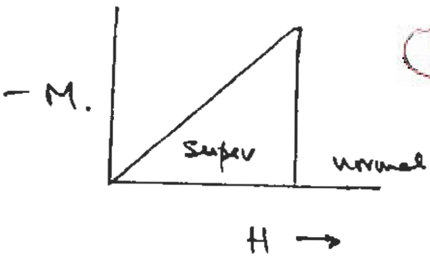
1

1

When sample is cooled below critical temperature it is superconducting it expels all lines of B.

In terms of M, H $\quad B = \mu_0(H + M)$

In superconductor $B = 0 \quad M = -H$. 1

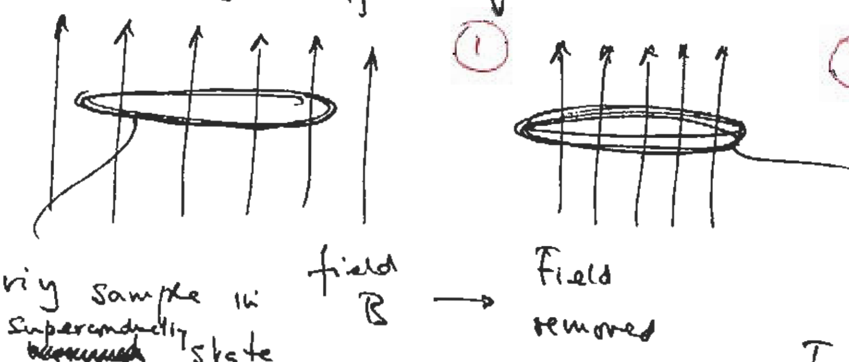


1 Graph of −M vs H for superconducting sample. 1

−M achieved by  1 superconducting currents flowing around surface.

5 of these.

(ii) Magnetic flux quantum



1 1

current flows in superconductor to conserve field that was present before.

Found that the superconducting current is quantised so that

the number of flux lines of B = Area of ring × B = $n\phi_0$ 1

where ϕ_0 is quantum of flux. and n is integer

Measurements determine $\phi_0 = \dfrac{h}{2e}$. 1   5 of these

BCS theory predicts $\phi_0 = \dfrac{h}{\text{unit of current carrier}}$. 1

Consistent that superconducti... 1

(iii) Superconductivity depends on condensation of electron pairs (Cooper pairs) into boson ground state. 1

However critical temperature is not set by Bose Condensation temperature (which is relatively high) but by interaction that causes electron pairing. 1 1

The critical temperature set by $kT_c \simeq$ pairing energy. 1

Pairing interaction caused by electron - lattice - electron interaction which lowers energy of $\uparrow\downarrow$ $k=0$ pairs. 1 1

5 of these

(iv) Experiments supporting the 2 fluid model of liquid He (II)

2 fluid model of liquid He (II) - interpenetrating 1
Superfluid and normal fluid -
Sets in at $T < T_\lambda$ 1
At $T = T_\lambda$ all normal - at $T \to 0$ all superfluid. 1

Expts - measurement of viscosity by timing torsional oscillation of discs immersed in liq He (II) 1
Normal fluid adheres to discs, increases moment of inertia & affects the oscillation time. - gives η_{discs}. 1

- also measure viscosity by flow through small capillary 1
- get $\eta_{capillary}$

Found $\eta_{capillary} \ll \eta_{discs}$ because $\eta_{capillary}$ sees superfluid 1

5 of these

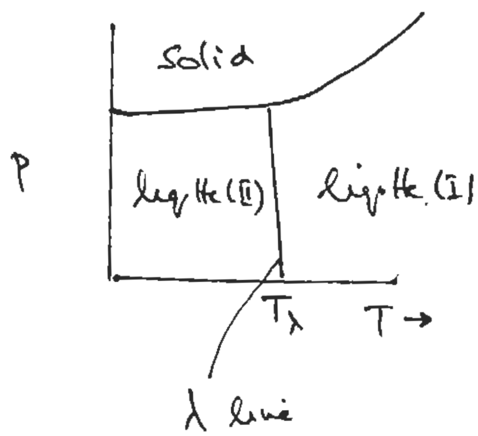
(v) The λ transition in liquid He^4.

Transition between liq He (I) and liq He (II) 1

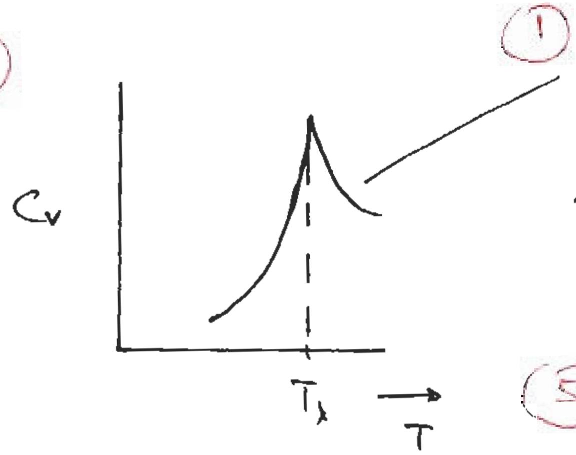
Occurs $\sim T_\lambda \sim 2.2$ K. 1

Internal energy U, and C_v much reduced in liquid He (II) 1
because of superfluid component in liquid He (II) and not 1
in liquid He (I)

Phase diagram 1



1



called λ transition from shape of C_v versus T graph.

5 of these

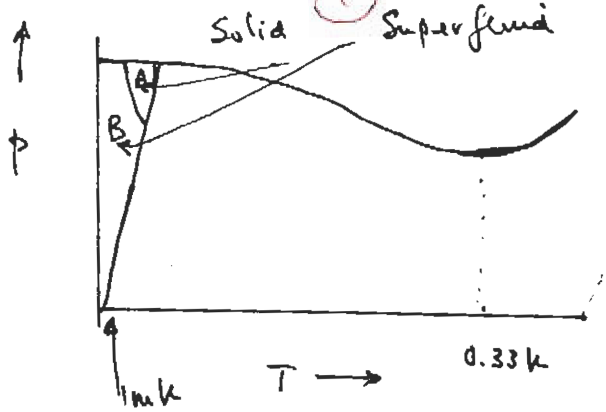
(vi) Superfluid He^3 — occurs T < 1 mK. 1

occurs via pairs of He^3 atoms. 1

which form bosons and undergo

Bose condensation. 1

Phase diagram



1

Pairs have S = 1 L = 1. 1

Complex behaviour - more than

one phase A, B. 1

Phases sensitive to magnetic field 1

5 of these

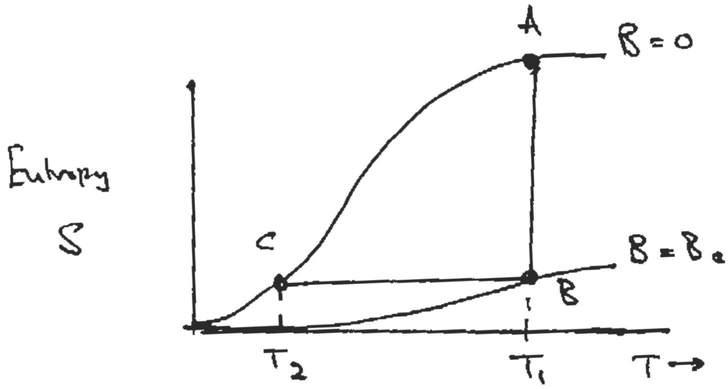
Adiabatic demagnetization.

Some indication that it will have magnetic moment

(vi) Sample – can be nuclear or atomic

Choose nuclear Cu metal Cu^{63} nucleus $I = 3/2$. (2)

(vii) Process.



1. Magnetise isothermally at temp T_1 (apply field with sample connected to heat bath at T_1. A → B (1)

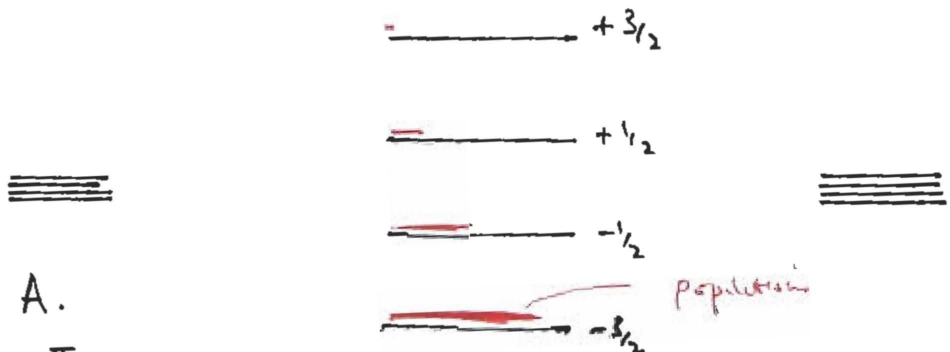
2. Isolate sample. (1)

3. Demagnetise adiabatically B → C. Sample temp drops to T_2. (1)

Process (3)

(viii) Theory

Theory (4)



(1) diagram

A.

Temp T_1
Substates nearly degenerate
Populations nearly same (1)

→ mag

B

field Ba applied at T_1
States split.
Populations heavily favour lower energy states (1)

→ demag

C

Adiabatic (S const)
So populations stay same.
These populations with small splitting → low temp T_2 (1)

(ix) Populations in B + C determined by B/T.

Calc 3

But populations same. – Thus $\left(\frac{B}{T}\right)_B = \left(\frac{B}{T}\right)_C$ (1)

Hence $\frac{10.0}{2\times10^{-3}} = \frac{5\times10^{-3}}{T_2}$ (1)

$T_2 = \frac{10\times10^{-6}}{10} = 1\times10^{-6}\,K$. (1) (12)

OK (3a)
Helium dilution refrigerator

(i) Form of helium – mixture of He^3 and He^4 liquids (1) + (1)

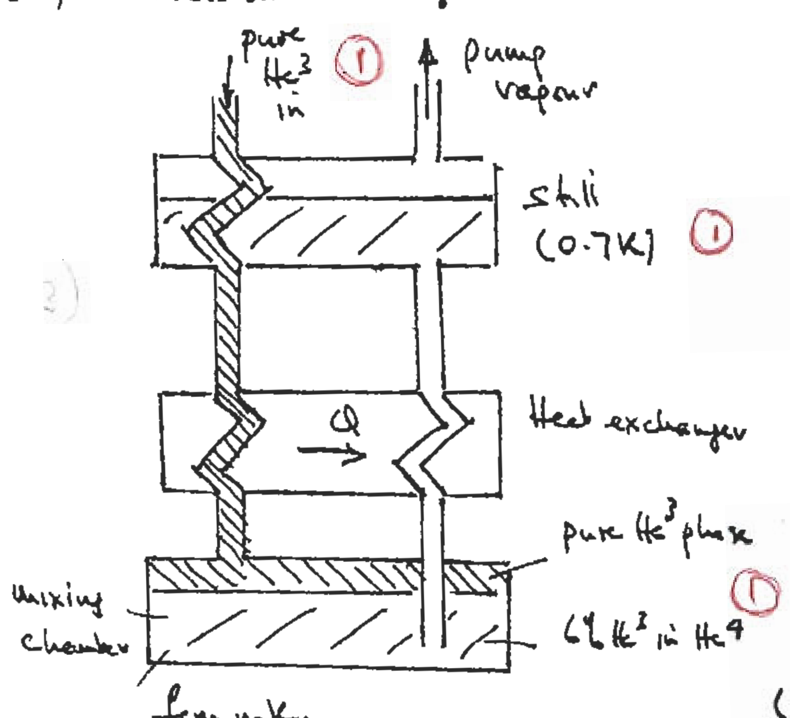
(ii) Theory of cooling

Below triple point $T = 0.86\,K$ He^3 and He^4 liquids form a 2 phase system (1) — He^3 rich phase
He^4 rich phase

At $T = 0$ He^3 rich phase $\to$ 100% He^3 (1)
He^4 rich phase $\to$ 6% He^3 + 94% He^4 (1)

Cooling.
In mixing chamber (see diagram) phase of He^3 floats on top of He^4 (6% He^3) phase (1)
Cooling occurs at phase boundary as He^3 atoms evaporate from He^3 rich to He^4 rich phases. (1) This takes energy (1) from He^3 rich phase and cools it. Theory any (4) of the (6)

(iii) Schematic diagram



(iv) Experimental process.
Apparatus works by allowing above evaporation to occur continuously. (in cycles)
Dilute phase pumped – pure He^3 gas comes off (1) and is circulated down thru' heat exchanger (1) to pure He^3 layer in mixing chamber – continuous process.

(v) (1) Starting temp ~ 0.5K

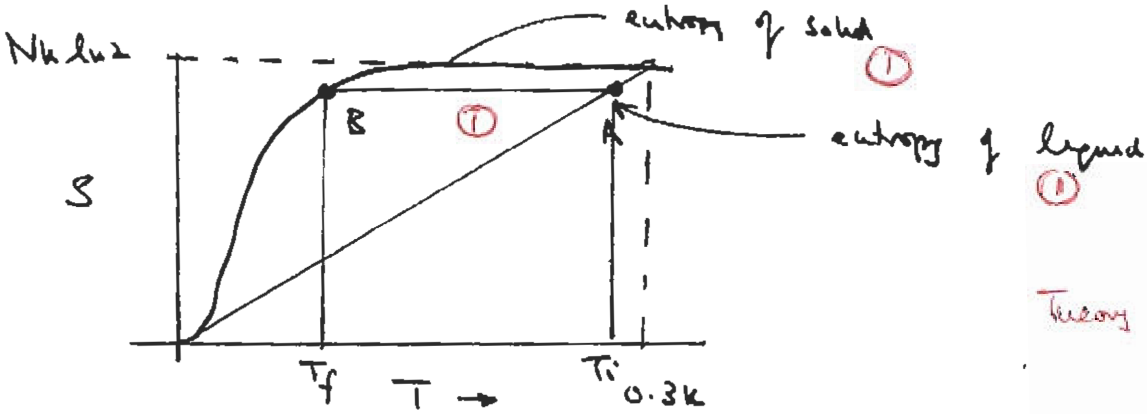
EITHER 5(a)

Pomeranchuk cooling.

Uses liquid / solid He^3. (1) + (1)

Theory. In temp range $0 \rightarrow 0.3K$ entropy S of solid He^3 is greater than liquid He^3. (1)

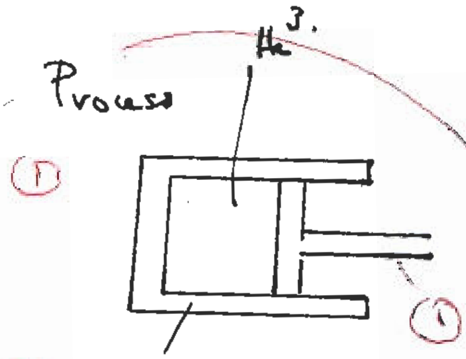
This because main entropy is in nuclear spins (1) — this greater in Boltzmann distrib^n in distinguishable atoms than in Fermi-Dirac distrib^n in indistinguishable liq atoms. (1)



Theory any (4) of this (1)

Property + theory sketch of S

Process



(1) Insulating walls

— compress liq $He^3 \longrightarrow$ solid He^3 (1)
adiabatically (const S).

— this reduces temp $T_i \longrightarrow T_f$. (1)

Basic procedure

In practice start ~ ~~...~~ 0.2 K (1)
and will end at ~10 mK. (1)

start + finish

One shot process. — connect sample to cell via conducting path (1)

schematic diagram

Type used He^3	(2)	
Theory + basic procedure	(6)	13.
Schematic diag	(3)	
Start + finish	(2)	

12 marks.